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Symbolical and geometrical characterizations of Kronecker sequences by using the accelerated Brun's algorithm

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Abstract

We give a van der Corput-type expression of the multi-dimensional Kronecker sequence and a geometrical characterization of its distribution property. We can consider the latter to be a multi-dimensional analogue of the "three-distance theorem." All these results were obtained by using the accelerated Brun's algorithm.

Key words: ergodic theory, irrational rotation, Kronecker sequence, low-discrepancy sequence, accelerated Brun's algorithm, three-distance theorem

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0 Introduction

The sequence $x_n = (n\alpha_1 - [n\alpha_1], \ldots, n\alpha_s - [n\alpha_s])$; $n \in \mathbb{N}$, $(\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$, called the Kronecker sequence with respect to $(\alpha_1, \ldots, \alpha_s)$, is distributed uniformly in the s-dimensional unit cube if and only if $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$. There is another well-known classical uniformly distributed sequence, called the van der Corput sequence. Many studies have been made of the distribution properties of these sequences [2,6,7,9].

In this paper, we study Kronecker sequences by using the accelerated Brun's algorithm [13].

Theorem 3.1 shows that we can construct the set of admissible words and the orbit of the origin by the adding machine transformation (Definition 3.5) on this set expresses the given Kronecker sequences. We can consider the van der Corput sequence to be the orbit of the origin under the adding machine transformation. Therefore, we say that the theorem gives a van der Corput-type expression of the Kronecker sequence. Following this principle of regarding the van der Corput sequence as an orbit of the adding machine transformation, a generalization of the van der Corput sequence is studied in [8,10,11]. Pagès [12] and Hellekalek [3] also consider the van der Corput sequence from this point of view.

We see from Theorem 5.1 that the distribution of the Kronecker sequence is connected with the stepped surface associated with the accelerated Brun's algorithm. The notion of stepped surfaces is introduced by Ito and Ohtsuki [5]. They construct the stepped surface associated with the modified Jacobi-Perron algorithm. The theorem gives a geometrical characterization of the Kronecker sequence and it is reasonable to say that this theorem is a multidimensional analogue of the classical three-distance theorem for the one-dimensional sequence generated by irrational rotation [14]. We emphasize here that the ergodic property of irrational rotations plays an essential role in the proof of the theorem.

1 Kronecker sequences

First, we recall the notions of irrational rotations and Kronecker sequences.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are the sets of all natural numbers, all integers, all rational numbers, and all real numbers, respectively. We also set

$$\mathbb{R}_{>a} = \{r \in \mathbb{R} \mid r > a\}$$
$$\mathbb{Z}_{\geq n} = \{i \in \mathbb{Z} \mid i \geq n\}$$
$$\vdots$$

and so on. I denotes the unit matrix. I_d denotes the d-dimensional unit matrix.

For $x \in \mathbb{R}$, $[x]$ denotes the integer part of x, and $[(x_1, \ldots, x_s)]$ means for $([x_1], \ldots, [x_s])$.

Let F_α be a parallel shift on $\mathbb{R}^s$ by α, where $\alpha \in \mathbb{R}^s$, that is to say $F_\alpha : x \mapsto x + \alpha$.

Definition 1.1 *A transformation on the s-dimensional unit cube* $[0, 1)^s$ *defined by* $x \mapsto F_\alpha x \pmod{\mathbb{Z}^s}$ *is called an irrational rotation if* $\alpha = (\alpha^1, \ldots, \alpha^s)$ *satisfies the following condition:*

(IR) $1, \alpha^1, \ldots, \alpha^s$ *are linearly independent over* $\mathbb{Q}$.

Let $\alpha = (\alpha^1, \ldots, \alpha^s) \in (0, 1)^s$ satisfy the condition **(IR)** and $\alpha' = (1 + \alpha^1 + \cdots + \alpha^s)^{-1}\alpha$. Let L_α be a $\mathbb{Z}$-module defined by (1.1).

$$L_\alpha = \mathbb{Z}\begin{pmatrix} 1+\alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^s \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \alpha^1 \\ 1+\alpha^2 \\ \vdots \\ \alpha^s \end{pmatrix} + \cdots + \mathbb{Z}\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ 1+\alpha^s \end{pmatrix} \tag{1.1}$$

The transformation $F_{\alpha'} \pmod{\mathbb{Z}^s}$ over $\mathbb{R}^s/\mathbb{Z}^s$ and the transformation $F_\alpha \pmod{L_\alpha}$ over $\mathbb{R}^s/L_\alpha$ are isomorphic, that is to say there exists a linear isomorphism $\Phi_\alpha : \mathbb{R}^s/L_\alpha \to \mathbb{R}^s/\mathbb{Z}^s$ that satisfies $\Phi_\alpha \circ (F_\alpha \pmod{L_\alpha}) = (F_{\alpha'} \pmod{\mathbb{Z}^s}) \circ \Phi_\alpha$. Note that when α satisfies **(IR)**, α' also satisfies **(IR)** and vice versa. In this paper, we consider $F_\alpha \pmod{L_\alpha}$ rather than $F_{\alpha'} \pmod{\mathbb{Z}^s}$.

Definition 1.2 *Let* $\alpha \in (0, 1)^s$ *satisfy* **(IR)** *and let* L_α *be a* $\mathbb{Z}$*-module defined by* (1.1). *We define the transformation* R_α *over* $\mathbb{R}^s/L_\alpha$ *as* $F_\alpha \pmod{L_\alpha}$. *The s-dimensional Kronecker sequence* $K_\alpha = \{K_\alpha(n)\}_{n=0}^{\infty}$ *with respect to* α *is defined by* $K_\alpha(0) = 0,\ K_\alpha(n + 1) = R_\alpha K_\alpha(n)$.

2 Accelerated Brun's algorithm

In this section, we define the multidimensional continued fraction algorithm, called the accelerated Brun's algorithm [13].

Definition 2.1 *Let*

$$X = \left\{ x = (x^1, \ldots, x^d) \in [0,1)^d \,\middle|\, x^1 > x^2 > \cdots > x^d \right\}.$$

For $\alpha = (\alpha^1, \ldots, \alpha^d) \in X$ that satisfies **(IR)**, *we define*

$$a(\alpha) = \left[\frac{1}{\alpha^1}\right],$$

$$\varepsilon(\alpha) = \begin{cases} 1 & \text{if } \frac{1}{\alpha^1} - \left[\frac{1}{\alpha^1}\right] > \frac{\alpha^2}{\alpha^1} \\ i & \text{if } \frac{\alpha^i}{\alpha^1} > \frac{1}{\alpha^1} - \left[\frac{1}{\alpha^1}\right] > \frac{\alpha^{i+1}}{\alpha^1} \quad \text{and} \quad 1 < i < d \\ d & \text{if } \frac{\alpha^d}{\alpha^1} > \frac{1}{\alpha^1} - \left[\frac{1}{\alpha^1}\right] \end{cases}$$

and

$$T(\alpha) = \left(\frac{\alpha^2}{\alpha^1}, \ldots, \frac{\alpha^{\varepsilon(\alpha)}}{\alpha^1}, \frac{1}{\alpha^1} - a(\alpha), \frac{\alpha^{\varepsilon(\alpha)+1}}{\alpha^1}, \ldots, \frac{\alpha^d}{\alpha^1} \right).$$

The triple $(X, T, (a(\alpha), \varepsilon(\alpha)))$ is called the accelerated Brun's algorithm. We also define $\alpha_n, a_n,$ and ε_n as follows:

$$\alpha_n = \begin{cases} \alpha & \text{if } n = 0 \\ T(\alpha_{n-1}) & \text{if } n \geq 1, \end{cases}$$

$$(a_n, \varepsilon_n) = (a(\alpha_{n-1}), \varepsilon(\alpha_{n-1})) \quad \text{for } n \geq 1.$$

Definition 2.2 *For $a \in \mathbb{N}$ and $\varepsilon \in \{1, \ldots, d\}$, we define a matrix $A_{(a,\varepsilon)} \in GL(d+1, \mathbb{Z})$ as follows:*

$$A_{(a,\varepsilon)} = (A_{ij})_{0 \leq i,j, \leq d}$$

where

$$A_{ij} = \begin{cases} a & \text{if } (i,j) = (0,0) \\ 1 & \text{if } (i,j) = (0,\varepsilon) \\ 1 & \text{if } (i,j) = (i,i-1) \text{ and } 1 \leq i \leq \varepsilon \\ 1 & \text{if } (i,j) = (i,i) \text{ and } \varepsilon + 1 \leq i \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3 *For $\alpha \in X$ satisfying* **(IR)**, *we define $M(\alpha)$, $M_n(\alpha)$ and*

$\theta_n(\alpha)$ as follows:

$$M(\alpha) = A_{(a(\alpha),\varepsilon(\alpha))},$$
$$M_n(\alpha) = \begin{cases} I & \text{if } n = 0 \\ M_{n-1}(\alpha)M(\alpha_{n-1}) & \text{if } n \geq 1, \end{cases}$$
$$\theta_n(\alpha) = \begin{cases} 1 & \text{if } n = 0 \\ \theta_{n-1}(\alpha)\alpha^1_{n-1} & \text{if } n \geq 0. \end{cases}$$

For $0 \leq i, j \leq d$, we define $m_n(\alpha; i, j)$ as the (i, j)-entry of $M_n(\alpha)$, that is to say

$$M_n(\alpha) = (m_n(\alpha; i, j))_{0 \leq i,j \leq d}\,.$$

We also define

$$l_n(\alpha, j) = \sum_{i=0}^{d} m_n(\alpha; i, j).$$

Definition 2.4 *For $\alpha \in X$ satisfying* **(IR)**, *we define $b_n(\alpha) \in \mathbb{R}^d$ and $d \times d$-matrices $B(\alpha), B_n(\alpha)$ as follows:*

$$b_n(\alpha) = \left(m_n(\alpha; 0, 0)\alpha^j - m_n(\alpha; j, 0)\right)_{1 \leq j \leq d} \quad \text{for } n \geq 0,$$
$$B(\alpha) = (B_{ij})_{1 \leq i,j \leq d}$$
$$\text{where} \quad B_{ij} = \begin{cases} 1 & \text{if } (i,j) = (i, i-1) \text{ and } 2 \leq i \leq \varepsilon(\alpha) \\ 1 & \text{if } (i,j) = (i,i) \text{ and } \varepsilon(\alpha) + 1 \leq i \leq d \\ -\alpha^i & \text{if } (i,j) = (i, \varepsilon(\alpha)) \\ 0 & \text{otherwise} \end{cases},$$

and

$$B_n(\alpha) = \begin{cases} I & \text{if } n = 0 \\ B_{n-1}(\alpha)B(\alpha_{n-1}) & \text{if } n \geq 1. \end{cases}$$

We also define $b^i_n(\alpha)$ as the i-th element of $b_n(\alpha)$, that is to say,

$$b_n(\alpha)^i = m_n(\alpha; 0, 0)\alpha^i - m_n(\alpha; i, 0) \quad i = 1, \ldots, d.$$

Definitions 2.2, 2.3, and 2.4 and direct calculations lead to the following proposition:

Proposition 2.1 *For all $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{pmatrix} 1 \\ {}^t\alpha \end{pmatrix} = \theta_n(\alpha) M_n(\alpha) \begin{pmatrix} 1 \\ {}^t\alpha_n \end{pmatrix} \tag{2.1}$$

and

$$b_n(\alpha) = B_n(\alpha)\,{}^t\alpha_n. \tag{2.2}$$

3 Van der Corput-type expression

We construct a unique expansion of natural numbers associated with the accelerated Brun's algorithm. By using this expansion, we obtain a van der Corput-type expression of the Kronecker sequence.

From now on, in this paper, we let $\alpha \in X$ satisfy **(IR)** and we let α_n and (a_n, ε_n) be generated from α by the accelerated Brun's algorithm.

From definition 2.3, we have

$$\begin{aligned} l_{n+1}(\alpha; 0) &= a_{n+1} l_n(\alpha; 0) + l_n(\alpha; 1) \\ l_{n+1}(\alpha; 1) &= l_n(\alpha; 2) \\ &\vdots \\ l_{n+1}(\alpha; \varepsilon_{n+1} - 1) &= l_n(\alpha; \varepsilon_{n+1}) \\ l_{n+1}(\alpha; \varepsilon_{n+1}) &= l_n(\alpha; 0) \\ l_{n+1}(\alpha; \varepsilon_{n+1} + 1) &= l_n(\alpha; \varepsilon_{n+1} + 1) \\ &\vdots \\ l_{n+1}(\alpha; d) &= l_n(\alpha; d). \end{aligned} \tag{3.1}$$

These recurrent relations (3.1) immediately lead us to the following proposition:

Proposition 3.1 *For all $n \geq 0$, there exists unique $k \geq 0$ which satisfies*

$$l_n(\alpha; 1) = l_{n-k}(\alpha; 0).$$

Definition 3.1 *We define $k(n)$ as the k in the previous proposition.*

For any positive integer N, we have the unique expansion of N in the following definition. This is a d-dimensional analogue of the well-known expansion associated with one-dimensional continued fraction [4,6,15].

Definition 3.2 *Let N be a positive integer. We define $c(N)$, $e(N)$, and $r(N)$ as follows:*

$$\begin{aligned} c(N) &= \max\{c \in \mathbb{Z}_{\geq 0} |\ l_c \leq N\} \\ e(N) &= \max\{e \in \mathbb{Z}_{\geq 0} |\ el_{c(N)} \leq N\} \\ r(N) &= N - e(N)l_{c(N)}, \end{aligned}$$

where $l_n = l_n(\alpha; 0)$. We remark that $r(N) < N$ holds and, taking this inequality into account, we can define $s(N)$ and c_j, $j = 0, \dots, s(N)$ thus:

$$\begin{aligned} s(N) &= \min\{s \in \mathbb{Z}_{\geq 0} |\ r^{s+1}(N) = 0\} \\ c_j &= \begin{cases} c(N) & \text{if } j = 0 \\ c(r^j(N)) & \text{if } j \geq 1, \end{cases} \end{aligned}$$

where $r^j(N)$ denotes $\underbrace{r(r(\dots r(N)\dots))}_{j \text{ times}}$. Define $e_j(N)$, $j = 0, 1, \dots, c(N)$ as follows:

$$e_j(N) = \begin{cases} e(r^i(N)) & \text{if } 0 \leq \exists i \leq s(N) \text{ such that } j = c_i, \\ 0 & \text{otherwise.} \end{cases}$$

We then have the unique expansion of N,

$$N = \sum_{j=0}^{c(N)} e_j(N) l_j(\alpha; 0), \tag{3.2}$$

associated with the sequence $\{l_n(\alpha; 0)\}_{n=0}^{\infty}$.

Definition 3.3 *We define a set Ω_α^0 of sequences of non-negative integers as follows:*

$$\Omega_\alpha^0 = \left\{(e_0(N), e_1(N), \dots, e_{c(N)}(N), 0, 0, \dots) \mid N = 0, 1, 2, \dots\right\}.$$

We also define $\Omega_\alpha^0(N)$ as $(e_0(N), \dots, e_{c(N)}(N), 0, 0, \dots)$.

Definition 3.4 *Let Y_i be a finite set $\{0, 1, \dots, a_i\}$ with discrete topology and $Y = \prod_{i=0}^{\infty} Y_i$ with the product topology. We define Ω_α as the closure of Ω_α^0 in Y.*

From Proposition 3.1, Definition 3.2, Definition 3.3, and Definition 3.4 we have the following proposition:

Proposition 3.2 $(e_0, e_1, \dots, e_n, \dots) \in (\mathbb{Z}_{\geq 0})^{\mathbb{N}}$ *belongs to Ω_α if and only if the following two conditions hold:*

(1) $e_j \le a_{j+1}$ *for all* $j = 0, 1, 2, \ldots$.
(2) $e_j = a_{j+1}$ *implies* $e_{j-1} = \cdots = e_{j-k(j)-1} = 0$

The adding machine transformation $1^+ : \Omega_\alpha \to \Omega_\alpha$ is defined as in the following definition.

Definition 3.5 *For* $e = (e_0, e_1, \ldots) \in \Omega_\alpha$ *we define*

$$1^+(e) = (0, \ldots, 0, e_j + 1, e_{j+1}, e_{j+2}, \ldots),$$

where

$$j = \min\{i \in \mathbb{Z}_{\ge 0} \mid (0, \ldots, 0, e_i + 1, e_{i+1}, e_{i+2}, \ldots) \in \Omega_\alpha\}.$$

We say $0 = (0, 0, \ldots)$ as the origin of Ω_α. Remark that

$$(1^+)^n(0) = \Omega_\alpha^0(n) \text{ for all } n \in \mathbb{Z}_{\ge 0}$$

holds.

In the following definition, we define a mapping ρ from Ω_α^0 to $\mathbb{R}^d/L_\alpha$.

Definition 3.6

$$\rho(e_0, e_1, \ldots, e_n, 0, 0 \ldots) = \sum_{k=0}^{n} e_k b_k(\alpha) \pmod{L_\alpha}$$

When $\sum_{k=0}^{\infty} e_k b_k(\alpha) \pmod{L_\alpha} = y \in \mathbb{R}^d/L_\alpha$ exists for an $e = (e_0, e_1, \ldots) \in \Omega_\alpha$, $\rho(e)$ denotes the value y and we say that e is the expansion of y.

The following theorem shows that the adding machine transformation on Ω_α and the irrational rotation R_α are connected by ρ.

Theorem 3.1 *Let* $e \in \Omega_\alpha$. *When* $\rho(e)$ *exists, it follows that*

$$\rho(1^+(e)) = R_\alpha(\rho(e)).$$

Corollary 3.1

$$K_\alpha(N) = \rho(\Omega_\alpha^0(N)) \quad \textit{for all } N \in \mathbb{Z}_{\ge 0}.$$

PROOF. Let $e = (e_0, e_1, \ldots)$ and $1^+(e) = (0, \ldots, 0, e_k + 1, e_{k+1}, e_{k+2}, \ldots)$. The following equality

$$l_k(\alpha; 0) = \sum_{j=0}^{k-1} e_j l_j(\alpha; 0) + 1 \tag{3.3}$$

holds from Definition 3.2. From Definition 2.3 and Definition 2.4, we have

$$
\begin{aligned}
l_j(\alpha;0)\alpha &= \sum_{i=0}^{d} m_j(\alpha;i,0)\alpha \\
&= b_j(\alpha) + m_j(\alpha;1,0)\begin{pmatrix}1+\alpha^1\\ \alpha^2\\ \vdots\\ \alpha^d\end{pmatrix} + m_j(\alpha;2,0)\begin{pmatrix}\alpha^1\\ 1+\alpha^2\\ \vdots\\ \alpha^d\end{pmatrix} \\
&\quad + \cdots + m_j(\alpha;d,0)\begin{pmatrix}\alpha^1\\ \alpha^2\\ \vdots\\ 1+\alpha^d\end{pmatrix} \\
&= b_j(\alpha) \pmod{L_\alpha} \qquad \text{for all} \quad j \in \mathbb{Z}_{\geq 0}.
\end{aligned} \tag{3.4}
$$

From (3.3),

$$
\begin{aligned}
b_k(\alpha) - \sum_{j=0}^{k-1} e_j b_j(\alpha) &= l_k(\alpha;0)\alpha - \sum_{j=0}^{k-1} e_j l_j(\alpha;0)\alpha \\
&= \left(l_k(\alpha;0) - \sum_{j=0}^{k-1} e_j l_j(\alpha;0) \right) \alpha \\
&= \alpha
\end{aligned} \tag{3.5}
$$

holds. We also have

$$
\rho(1^+(e)) - \rho(e) = b_k(\alpha) - \sum_{j=0}^{k-1} e_j b_j(\alpha) \pmod{L_\alpha} \tag{3.6}
$$

from Definition 3.6. From these equalities (3.5) and (3.6), we have

$$
\rho(1^+(e)) = \rho(e) + \alpha \pmod{L_\alpha},
$$

and the theorem follows. □

4 Stepped surfaces

We define stepped surfaces and substitutions on these from the accelerated Brun's algorithm. The notion of stepped surfaces was first introduced by Ito and Ohtsuki [5]. We introduce the notion following Arnoux and Ito [1].

Let $\mathcal{A}$ be a set of $d+1$ letters, that is to say $\mathcal{A} = \{0, 1, \ldots, d\}$. Let $\mathcal{A}^*$ be the set of finite words on the set $\mathcal{A}$, that is to say $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$. $\mathcal{A}^*$ is endowed with the concatenation product. A substitution σ on $\mathcal{A}^*$ is an endomorphism of $\mathcal{A}^*$ defined as follows:

(1) for $i \in \mathcal{A}$, $\sigma(i) = W^{(i)} \in \mathcal{A}^*$,
(2) for all $U, V \in \mathcal{A}^*$, $\sigma(UV) = \sigma(U)\sigma(V)$.

For a word U, leng(U) denotes its length and $U(i) \in \mathcal{A}$ denotes the i-th letter of U, that is to say $U = U(1)U(2)\cdots U(\text{leng}(U))$. When $j < k$, $U[j,k)$ denotes the word $U(j)U(j+1)\cdots U(k-1)$ and $U(j,k]$ the word $U(j+1)U(j+2)\cdots U(k)$. When $j \geq k$, $U[j,k)$ and $U(j,k]$ denote the empty word.

Definition 4.1 *We define a homomorphism $f : \mathcal{A} \to \mathbb{Z}^{d+1}$ as follows:*

$$f(i) = \mathbf{e}_i \quad \textit{for } i \in \mathcal{A},$$
$$f(UV) = f(U) + f(V) \quad \textit{for } U, V \in \mathcal{A}^*,$$

where $\mathbf{e}_i$, $i = 0, 1, \ldots, d$ denotes the i-th unit vector in $\mathbb{R}^{d+1}$. We define a linear transformation ${}^0\sigma$ by the following commutative relation:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ f\big\downarrow & & \big\downarrow f \\ \mathbb{Z}^{d+1} & \xrightarrow{{}^0\sigma} & \mathbb{Z}^{d+1} \end{array}$$

Definition 4.2 *A substitution σ is called unimodular if ${}^0\sigma$ has determinant* 1 *or* -1.

We consider only unimodular substitutions in the following part of this paper.

Definition 4.3 *We define $\mathbb{Z}$-modules $\mathcal{F}$ and $\mathcal{F}^*$ as follows:*

$$\mathcal{F} = \bigoplus_{\mathbb{Z}^{d+1} \times \mathcal{A}} \mathbb{Z}$$
$$\mathcal{F}^* = \{u \in \text{Hom}_{\mathbb{Z}}(\mathcal{F}, \mathbb{Z}) \mid \textit{support of } u \textit{ is finite.}\}$$

For $g \in \mathcal{F}$ and $h \in \mathcal{F}^*$, $\langle h, g \rangle$ denotes the natural pairing.

For $x \in \mathbb{Z}^{d+1}$ and $i \in \mathcal{A}$, (x, i) is identified with the element of $\mathcal{F}$ which takes value 1 at (x, i), and 0 elsewhere. (x, i^*), $i \in \mathcal{A}$ denotes the element of $\mathcal{F}^*$ defined by

$$\langle (x, i^*), (y, j) \rangle = \begin{cases} 1 & \text{if } x = y \text{ and } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the set $\left\{(x,i) \,\middle|\, x \in \mathbb{Z}^{d+1}, i \in \mathcal{A}\right\}$ is a basis of $\mathcal{F}$ and the set $\left\{(x,i^*) \,\middle|\, x \in \mathbb{Z}^{d+1}, i \in \mathcal{A}\right\}$ is its dual basis. For a unimodular substitution σ defined as above, we define the one dimensional geometric realization ${}^1\sigma : \mathcal{F} \to \mathcal{F}$ and its dual map ${}^1\sigma^* : \mathcal{F}^* \to \mathcal{F}^*$ as in the following definition.

Definition 4.4

$$ {}^1\sigma(x,i) = \sum_{k=1}^{\mathrm{leng}(W^{(i)})} \left({}^0\sigma(x) + f\left(W^{(i)}[1,k)\right), W^{(i)}(k)\right) $$

$$ \langle {}^1\sigma^*(u), v\rangle = \langle u, {}^1\sigma(v)\rangle, \quad \textit{for all } u \in \mathcal{F}^*, v \in \mathcal{F}. $$

From Definition 4.3, Definition 4.2, and Definition 4.4, we have the following lemma.

Lemma 4.1 (Arnoux-Ito [1]) *The map ${}^1\sigma^*$ is defined by*

$$ {}^1\sigma^*(x,i^*) = \sum_{\substack{1\le k,\ j\in\mathcal{A}\\ W^{(j)}(k)=i}} \left({}^0\sigma^{-1}\left(x - f\left(W^{(j)}[1,k)\right)\right), j^*\right). $$

We define a mapping $\mathcal{R}$ that give a geometric interpretation of $\mathcal{F}^*$ and ${}^1\sigma^*$. For $i \in \mathcal{A}$, $t \in \mathbb{R}$ and $s \in \{0,1\}$, $\lambda^s(i;t)$ is defined as follows:

$$ \lambda^0(i;t) = t\mathbf{e}_i $$
$$ \lambda^1(i;t) = (1-t)\mathbf{e}_i. $$

We define E_i a subset of $\mathbb{R}^{d+1}$ as follows:

$$ E\begin{pmatrix} s_1, & s_2, & \ldots, & s_n \\ i_1, & i_2, & \ldots, & i_n \end{pmatrix} = \left\{\lambda^{s_1}(i_1;t_1) + \cdots + \lambda^{s_n}(i_n;t_n) \mid 0 \le t_1, \ldots, t_n < 1\right\}, $$

$$ E_i = \mathbf{e}_i + E\begin{pmatrix} 1, & 1, & \ldots, & 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & i-1, & i+1, & \ldots, & d \end{pmatrix}. $$

Definition 4.5 *Any $u \in \mathcal{F}^*$ is uniquely expressed as a finite sum*

$$ u = \sum_k u_k(x_k, i_k^*) \textit{ if } k \ne j,\ (x_k, i_k) \ne (x_j, i_j). $$

When all coefficients u_k are 1 or -1, u is called geometric. For a geometric

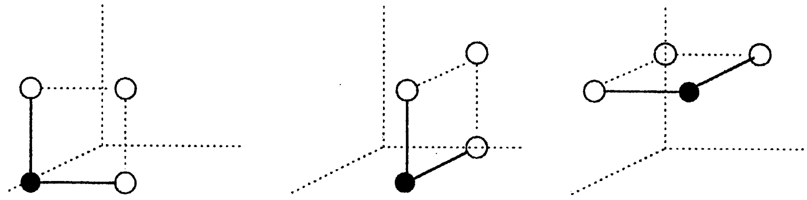

Fig. 4.1. E_0, E_1, and E_2, $(d = 2)$

$u \in \mathcal{F}^*$, $\mathcal{R}(u) \subset \mathbb{R}^{d+1}$ *is defined as follows:*

$$\mathcal{R}((x, i^*)) = x + E_i,$$
$$\mathcal{R}\left(\sum_k u_k(x_k, i_k^*)\right) = \bigsqcup_k \mathcal{R}((x_k, i_k^*)).$$

$\mathbf{v}_n \in \mathbb{R}^{d+1}$, $P_n, P_n^>, P_n^\geq \subset \mathbb{R}^{d+1}$ and $\mathcal{L}_n \subset \mathbb{Z}^{d+1}$ are defined for α as follows:

$$\mathbf{v}_n = \begin{cases} \sum_{k=0}^d \mathbf{e}_k & \text{if } n = 0 \\ {}^t\!A_{(a_n,\varepsilon_n)}\mathbf{v}_{n-1} & \text{if } n \geq 1, \end{cases} \tag{4.1}$$

$$P_n = \left\{x \in \mathbb{R}^{d+1} \,\middle|\, {}^t\!x\mathbf{v}_n = 0\right\}, \tag{4.2}$$

$$P_n^> = \left\{x \in \mathbb{R}^{d+1} \,\middle|\, {}^t\!x\mathbf{v}_n > 0\right\}, \tag{4.3}$$

$$P_n^\geq = \left\{x \in \mathbb{R}^{d+1} \,\middle|\, {}^t\!x\mathbf{v}_n \geq 0\right\}, \tag{4.4}$$

$$\mathcal{L}_n = \begin{cases} \mathbb{Z}(\mathbf{e}_1 - \mathbf{e}_0) + \mathbb{Z}(\mathbf{e}_2 - \mathbf{e}_0) + \cdots + \mathbb{Z}(\mathbf{e}_d - \mathbf{e}_0) & \text{if } n = 0 \\ A_{(a_n,\varepsilon_n)}^{-1}\mathcal{L}_{n-1} & \text{if } n \geq 1. \end{cases} \tag{4.5}$$

We have the following proposition immediately from the definition:

Proposition 4.1 *For all $n \in \mathbb{Z}_{\geq 0}$, $A_{(a_n,\varepsilon_n)}^{-1}P_{n-1} = P_n$ and $\mathcal{L}_n = P_n \cap \mathbb{Z}^{d+1}$ hold.*

In the following definition, we define stepped surfaces S_n and S_n' on P_n:

Definition 4.6 *$\mathcal{C}_n$ and $\mathcal{C}_n' \subset \mathcal{F}^*$ are defined as follows:*

$$\mathcal{C}_n = \left\{(x, i^*) \,\middle|\, \overline{\mathcal{R}((x, i^*))} \subset P_n^> \text{ and } x \notin P_n^>\right\}$$
$$\mathcal{C}_n' = \left\{(x, i^*) \,\middle|\, \overline{\mathcal{R}((x, i^*))} \subset P_n^\geq \text{ and } x \notin P_n^\geq\right\}.$$

$\mathcal{S}_n$ and $\mathcal{S}_n'$ are families of all finite subsets of $\mathcal{C}_n$ and $\mathcal{C}_n'$ respectively; that is,

$$\mathcal{S}_n = \left\{\sum_{\lambda \in \Lambda}(x_\lambda, i_\lambda^*) \,\middle|\, \begin{array}{c} \#\Lambda < \infty, (x_\lambda, i_\lambda^*) \in \mathcal{C}_n, \\ (x_\lambda, i_\lambda^*) \neq (x_{\lambda'}, i_{\lambda'}^*) \text{ for all } \lambda \neq \lambda' \end{array}\right\}$$

and

$$S'_n = \left\{ \sum_{\lambda\in\Lambda}(x_\lambda, i^*_\lambda) \;\middle|\; \begin{array}{c} \#\Lambda < \infty, (x_\lambda, i^*_\lambda) \in C'_n, \\ (x_\lambda, i^*_\lambda) \neq (x_{\lambda'}, i^*_{\lambda'}) \text{ for all } \lambda \neq \lambda' \end{array} \right\}$$

where elements are denoted as formal sums. An element of $\mathcal{S}_n$ *(resp.* $\mathcal{S}'_n$*) and its image by* $\mathcal{R}$ *are called a patch of the stepped surface. If* $U, V, W \in \mathcal{S}_n$ *(or* $\mathcal{S}'_n$*) satisfy* $U = V + W$*, we write* $V \prec U$ *and define* $U - V = W$*.* S_n *and* S'_n *are defined as follows:*

$$S_n = \bigcup_{U\in\mathcal{S}_n} \mathcal{R}(U), \quad S'_n = \bigcup_{U\in\mathcal{S}'_n} \mathcal{R}(U).$$

We now construct the substitution between stepped surfaces. First, we define substitutions σ_n, $n \in \mathbb{N}$ on the set $\mathcal{A}^*$ as follows:

$$\sigma_n(i) = \begin{cases} W_n^{(0)} = \underbrace{00\ldots0}_{a_n \text{ times}}1 & \text{if } i = 0 \\ W_n^{(1)} = 2 & \text{if } i = 1 \\ \vdots & \\ W_n^{(\varepsilon_n-1)} = \varepsilon_n & \text{if } i = \varepsilon_n - 1 \\ W_n^{(\varepsilon_n)} = 0 & \text{if } i = \varepsilon_n \\ W_n^{(\varepsilon_n+1)} = \varepsilon_n + 1 & \text{if } i = \varepsilon_n + 1 \\ \vdots & \\ W_n^{(d)} = d & \text{if } i = d. \end{cases} \tag{4.6}$$

We introduce the following two lemmas which show that ${}^1\sigma_n^*$ induces a mapping from $\mathcal{S}_{n-1}$ (resp. $\mathcal{S}'_{n-1}$) to $\mathcal{S}_n$ (resp. $\mathcal{S}'_n$).

Lemma 4.2 *For all* $(x, i^*) \in \mathcal{C}_{n-1}$ *(resp.* $\mathcal{C}'_{n-1}$*), it follows that* ${}^1\sigma_n^*(x, i^*) \in \mathcal{S}_n$ *(resp.* $\mathcal{S}'_n$*).*

PROOF. We prove the lemma for the case in which $(x, i^*) \in \mathcal{C}_{n-1}$. We can prove the case of $\mathcal{C}'_{n-1}$ in the same way. First we show that

$$\left(A^{-1}_{(a_n,\varepsilon_n)}\left(x - f\left(W_n^{(j)}[1, m)\right)\right), j^*\right) \in \mathcal{C}_n \tag{4.7}$$

holds for all i, m, j that satisfy $W_n^{(j)}(m) = i$.

Let i, m, j satisfy $W_n^{(j)}(m) = i$. We know by (4.1) that

$$\mathbf{v}_n = {}^tM_n(\alpha)\sum_{k=0}^{d} \mathbf{e}_k = (l_n(\alpha; 0), \ldots, l_n(\alpha, d)) \in \mathbb{N}^{d+1} \tag{4.8}$$

holds for all $n \in \mathbb{Z}_{\geq 0}$. From this fact and Definition 4.6, we have

$$A^{-1}_{(a_n,\varepsilon_n)}\left(x - f\left(W_n^{(j)}[1,m)\right)\right) \in P_n^{<}. \tag{4.9}$$

Taking Definition 4.1 into account, we see that

$$\begin{aligned} &{}^0\sigma_n^{-1}\left(x - f\left(W_n^{(j)}[1,k)\right)\right) + \mathbf{e}_j \\ &= {}^0\sigma_n^{-1}\left(x - f\left(W_n^{(j)}[1,k)\right) + f(\sigma(j))\right) \\ &= {}^0\sigma_n^{-1}\left(x + \mathbf{e}_i + f\left(W_n^{(j)}(k, \mathrm{leng}(W_n^{(j)})]\right)\right) \end{aligned} \tag{4.10}$$

holds. By (4.7) it follows that

$${}^t f\left(W_n^{(j)}(k, \mathrm{leng}(W_n^{(j)})]\right) \mathbf{v}_{n-1} > 0,$$

and it follows that

$$x + \mathbf{e}_i + {}^t f\left(W_n^{(j)}(k, \mathrm{leng}(W_n^{(j)})]\right) \in P_{n-1}^{>}. \tag{4.11}$$

From (4.10) and (4.11), we have

$${}^0\sigma_n^{-1}\left(x - f\left(W_n^{(j)}\right)\right) + \mathbf{e}_j \in P_n^{>}. \tag{4.12}$$

(4.7) follows from (4.9) and (4.12).

Second, we show that ${}^1\sigma_n^*(x, i^*)$ is geometric for all $(x, i^*) \in \mathcal{C}_{n-1}$. Let

$$\left({}^0\sigma_n^{-1}\left(x - f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) = \left({}^0\sigma_n^{-1}\left(x - f\left(W_n^{(j')}[1,k')\right)\right), j'^*\right).$$

If $W_n^{(j)}(k) = W_n^{(j')}(k) = i$ then $j = j'$. If $k < k'$ then $f(W_n^{(j')}[1,k')) \neq f(W_n^{(j)}[1,k))$ holds and it contradicts the fact that ${}^0\sigma_n \in GL(d+1;\mathbb{Z})$. Thus $k = k', j = j'$ holds. By virtue of Lemma 4.1 we know that ${}^1\sigma_n^*(x, i^*)$ is geometric. From this fact, (4.7), and Lemma 4.1, we see that the lemma is proved. □

Lemma 4.3 *For all $(x_1, j_1^*), (x_2, j_2^*) \in \mathcal{C}_{n-1}$ (or $\mathcal{C}'_{n-1}$), if there exists a unit chip (y, i^*) which satisfies $(y, i^*) \prec {}^1\sigma_n^*(x_1, j_1^*)$ and $(y, i^*) \prec {}^1\sigma_n^*(x_2, j_2^*)$ then $(x_1, j_1^*) = (x_2, j_2^*)$ holds.*

PROOF. We prove the lemma for the case of $\mathcal{C}_{n-1}$. We can prove the case of $\mathcal{C}'_{n-1}$ in the same way. Let

$$\begin{aligned} &(x_1, j_1^*), (x_2, j_2^*) \in \mathcal{C}_{n-1}, \\ &(y, i^*) \prec {}^1\sigma_n^*(x_1, j_1^*), \quad \text{and} \quad (y, i^*) \prec {}^1\sigma_n^*(x_2, j_2^*). \end{aligned} \tag{4.13}$$

By virtue of Lemma 4.1, we know that there exist k_1 and k_2 which satisfy the following:

$$(4.14) \qquad y = {}^0\sigma_n^{-1}\left(x_1 - f\left(W_n^{(i)}[1,k_1)\right)\right) = {}^0\sigma_n^{-1}\left(x_2 - f\left(W_n^{(i)}[1,k_2)\right)\right).$$

Remark that

$$(4.15) \qquad x_1 - f\left(W_n^{(i)}[1,k_1)\right) = x_2 - f\left(W_n^{(i)}[1,k_2)\right)$$

follows.

First, we assume $x_1 = x_2$. In this case, $k_1 = k_2$ and $W_n^{(j_1)}(k_1) = W_n^{(j_2)}(k_2) = i$ hold from (4.14). Then we see $j_1 = j_2$ follows from the definition of σ_n (4.6) and the lemma follows.

Then we assume $x_1 \neq x_2$. $k_1 \neq k_2$ holds. Let $k_1 < k_2$. We have $x_1 + f(W_n^{(i)}[k_1,k_2)) = x_2$ from (4.15). This equality and $j_1 = W_n^{(i)}(k_1)$ lead to $(x_2, j_2^*) \notin \mathcal{C}_{n-1}$. This contradicts (4.13). $\square$

Definition 4.7 *Mappings ${}^1\sigma_n^*|_{\mathcal{S}_n}$ and ${}^1\sigma_n^*|_{\mathcal{S}'_n}$ are called substitutions on the stepped surfaces associated with the accelerated Brun's algorithm.*

5 Kronecker sequences and domain exchange transformations

In this section, we see the correspondence between the distribution of the Kronecker sequence K_α and stepped surfaces associated with the accelerated Brun's algorithm.

First, we prepare five lemmas related to stepped surfaces and domain exchange transformations.

Let U, U', $U(i)$, and $U'(i)$ $(i \in \mathcal{A})$ denote elements of $\mathcal{F}^*$ as follows:

$$(5.1) \qquad \begin{aligned} U(i) &= (0, i^*), \quad U'(i) = (-\mathbf{e}_i, i^*), \\ U &= \sum_{i=0}^{d} U(i), \quad U' = \sum_{i=0}^{d} U'(i). \end{aligned}$$

From **(IR)**, U (resp. U') belongs to $\mathcal{S}_0$ (resp. $\mathcal{S}'_0$). Taking this into account, we define $U_n(i), U_n \in \mathcal{S}_n$ and $U'_n(i), U'_n \in \mathcal{S}'_n$ $(i \in \mathcal{A},\ n \in \mathbb{Z}_{\geq 0})$ as follows:

$$(5.2)\qquad \begin{aligned} U_n(i) &= \begin{cases} U(i) & \text{if } n = 0 \\ {}^1\sigma_n^*(U_{n-1}(i)) & \text{otherwise,} \end{cases} \\ U_n &= \begin{cases} U & \text{if } n = 0 \\ {}^1\sigma_n^*(U_{n-1}) & \text{otherwise,} \end{cases} \\ U'_n(i) &= \begin{cases} U'(i) & \text{if } n = 0 \\ {}^1\sigma_n^*(U'_{n-1}(i)) & \text{otherwise,} \end{cases} \\ U'_n &= \begin{cases} U' & \text{if } n = 0 \\ {}^1\sigma_n^*(U'_{n-1}) & \text{otherwise.} \end{cases} \end{aligned}$$

From Lemma 4.3,

$$(5.3)\qquad U_n = \bigsqcup_{i \in \mathcal{A}} U_n(i) \quad \text{and} \quad U'_n = \bigsqcup_{i \in \mathcal{A}} U'_n(i)$$

hold.

Figures 5.1,5.2, and 5.3 show examples of U, U', U_n, and U'_n in the case where $d = 2$, $\alpha^1 = 1/\sqrt{3}$, $\alpha^2 = 1/\sqrt{5}$, $(a_n, \varepsilon_n) = (1,2), (1,2), (1,2), (3,1), (4,2), (1,1), (3,1), (4,1), \ldots$. In these figures, we mark cells that belong to $U_n(2)$ or $U'_n(2)$ with crosses and cells that belong to $U_n(3)$ or $U'_n(3)$ with black squares.

Lemma 5.1 ([1]) *For all $n \in \mathbb{Z}_{\geq 0}$, $U \prec U_n$, $U' \prec U'_n$ and $U_n - U = U'_n - U'$ hold.*

PROOF. If $n = 0$, the statement holds. We assume that $U \prec U_{n-1}$, $U' \prec U'_{n-1}$ and $U_{n-1} - U = U'_{n-1} - U'$ hold. From these assumptions and following two equalities:

$$U_n - U = {}^1\sigma_n^*(U_{n-1} - U) + {}^1\sigma_n^*(U) - U$$

and

$$U'_n - U' = {}^1\sigma_n^*(U'_{n-1} - U') + {}^1\sigma_n^*(U') - U',$$

it is enough to show that $U \prec {}^1\sigma_n^*(U)$, $U' \prec {}^1\sigma_n^*(U')$ and ${}^1\sigma_n^*(U) - U = {}^1\sigma_n^*(U') - U'$ hold. If $W_n^{(j)}(1) = i$, we have $(0, j^*) \prec {}^1\sigma_n^*(0, i^*)$ from Lemma 4.1. Then $U \prec {}^1\sigma_n^*(U)$ holds. If $W_n^{(j)}(\mathrm{leng}(W_n^{(j)})) = i$, we have

$$\begin{aligned} {}^1\sigma_n^*(-\mathbf{e}_i, i^*) &\succ \left({}^0\sigma_n^{-1}\left(-\mathbf{e}_i - f\left(W_n^{(j)}\left[1, \mathrm{leng}\left(W_n^{(j)}\right)\right)\right)\right), j^*\right) \\ &= \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}\right)\right), j^*\right) \\ &= (-\mathbf{e}_j, j^*) \end{aligned}$$

from Lemma 4.1. Then $U' \prec {}^1\sigma_n^*(U')$ holds. When $W_n^{(j)}(k) = i$, we have

$$\mathbf{e}_i + f\left(W_n^{(j)}[1,k)\right) = f\left(W_n^{(j)}[1,k+1)\right) \quad \text{when} \quad k < \text{leng}\left(W_n^{(J)}\right) \tag{5.4}$$

and

$$ {}^0\sigma_n^{-1}\left(-\mathbf{e}_i - f\left(W_n^{(j)}[1,k)\right)\right) = -\mathbf{e}_j \quad \text{when} \quad k = \text{leng}\left(W_n^{(J)}\right). \tag{5.5}$$

From Lemma 4.1 and equalities (5.4) and (5.5), we have

(5.6)
$$\begin{aligned}
{}^1\sigma_n^*(U') &= \sum_{i\in\mathcal{A}} {}^1\sigma_n^*(-\mathbf{e}_i, i^*) \\
&= \sum_{i\in\mathcal{A}} \sum_{\substack{1\le k,\ j\in\mathcal{A} \\ W_n^{(j)}(k)=i}} \left({}^0\sigma_n^{-1}\left(-\mathbf{e}_i - f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) \\
&= \sum_{j\in\mathcal{A}} \sum_{k=1}^{\text{leng}(W_n^{(j)})} \left({}^0\sigma_0^{-1}\left(-\mathbf{e}_{W_n^{(j)}(k)} - f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) \\
&= \sum_{j\in\mathcal{A}} \sum_{k=1}^{\text{leng}(W_n^{(j)})-1} \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}[1,k+1)\right)\right), j^*\right) + \sum_{j\in\mathcal{A}}(-\mathbf{e}_j, j^*) \\
&= \sum_{j\in\mathcal{A}} \sum_{k=2}^{\text{leng}(W_n^{(j)})} \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) + U'
\end{aligned}$$

and

$$\begin{aligned}
{}^1\sigma_n^*(U) &= \sum_{i\in\mathcal{A}} {}^1\sigma_n^*(0, i^*) \\
&= \sum_{i\in\mathcal{A}} \sum_{\substack{1\le k,\ j\in\mathcal{A} \\ W_n^{(j)}(k)=i}} \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) \\
&= \sum_{j\in\mathcal{A}} \sum_{k=2}^{\text{leng}(W_n^{(j)})} \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) + \sum_{j\in\mathcal{A}}(0, j^*) \\
&= \sum_{j\in\mathcal{A}} \sum_{k=2}^{\text{leng}(W_n^{(j)})} \left({}^0\sigma_n^{-1}\left(-f\left(W_n^{(j)}[1,k)\right)\right), j^*\right) + U.
\end{aligned} \tag{5.7}$$

From (5.6) and (5.7), ${}^1\sigma_n^*(U') - U' = {}^1\sigma_n^*(U) - U$ holds. □

Let π_n be a projection from $\mathbb{R}^{d+1}$ to $\{(x^0, x^1, \ldots, x^d) \in \mathbb{R}^{d+1} | \ x_0 = 0\}$ along ${}^t(1, \alpha)$. Hereafter, we identify $\mathbb{R}^d$ with $\{(x^0, x^1, \ldots, x^d) \in \mathbb{R}^{d+1} | \ x_0 = 0\}$. We

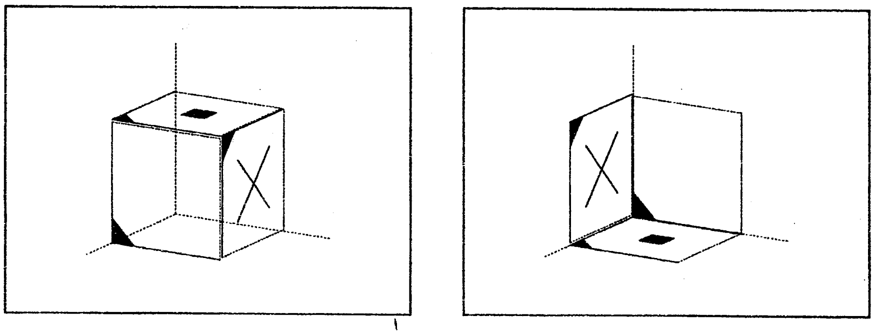

Fig. 5.1. U and U'

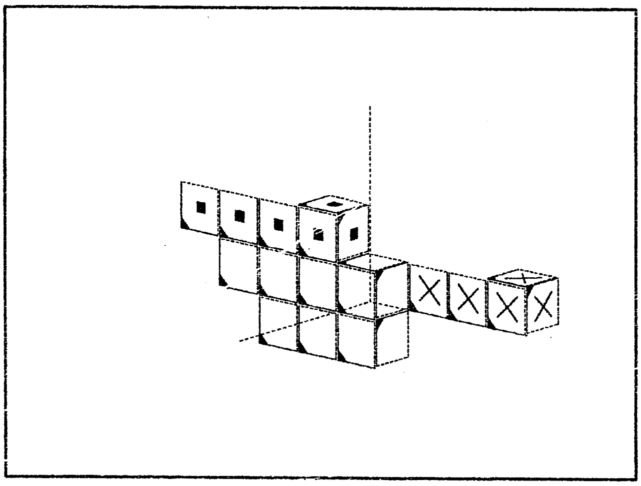

Fig. 5.2. U_4

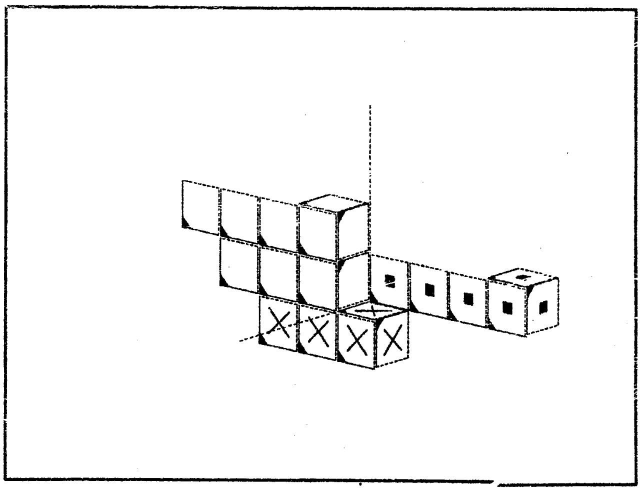

Fig. 5.3. U_4'

define D_n, D'_n, $D_n(i)$, and $D'_n(i)$ ($i \in \mathcal{A}$, $n \in \mathbb{Z}_{\geq 0}$) as follows:

$$D_n = \pi_n(\mathcal{R}(U_n)), \qquad D'_n = \pi_n(\mathcal{R}(U'_n)), \tag{5.8}$$
$$D_n(i) = \pi_n(\mathcal{R}(U_n(i))), \quad D'_n(i) = \pi_n(\mathcal{R}(U'_n(i))). \tag{5.9}$$

We also define $D = D_0$, $D' = D'_0$, $D(i) = D_0(i)$, and $D'(i) = D'_0(i)$. From (5.8) and Lemma 4.3, we have

$$\begin{aligned} &D_n = \bigsqcup_{i \in \mathcal{A}} D_n(i), \quad D'_n = \bigsqcup_{i \in \mathcal{A}} D'_n(i); \\ &\text{if } i \neq j \text{ then } D_n(i) \cap D_n(j) = D'_n(i) \cap D'_n(j) = \emptyset. \end{aligned} \tag{5.10}$$

It is trivial to show that $D = D'$. Then, from Lemma 5.1

$$D_n = D'_n \qquad \text{for all } n \in \mathbb{Z}_{\geq 0} \tag{5.11}$$

holds.

We introduce the domain exchange transformation on stepped surfaces and their images projected by π_n. We define $f_n(i) \in \mathbb{Z}^{d+1}$, $i \in \mathcal{A}$ as follows:

$$M_n(\alpha)^{-1} = \begin{pmatrix} f_n(0) & f_n(1) & \cdots & f_n(d) \end{pmatrix}, \tag{5.12}$$

that is to say $f_n(j)$ is the j-th column vector of $M_n(\alpha)^{-1}$. $f_n(j)^k$ denotes the k-th element of $f_n(j)$, that is to say $f_n(j) = {}^t(f_n(j)^0, \ldots, f_n(j)^d)$.

For $y \in \mathbb{Z}^{d+1}$ and $u = \sum_k u_k(x_k, i_k^*) \in \mathcal{F}^*$, we define $y + u$ as $\sum_k u_k(y + x_k, i_k^*)$.

Lemma 5.2 *For $i \in \mathcal{A}$, $U_n(i) = U'_n(i) + f_n(i)$ holds.*

Corollary 5.1 *For $i \in \mathcal{A}$, $D_n(i) = D'_n(i) + \pi_n f_n(i)$ holds.*

PROOF. If $n = 0$,

$$U_0(i) = (0, i^*) = -\mathbf{e}_i + (-\mathbf{e}_i, i^*) = U'_0(i) + f_0(i)$$

holds and the statement follows. We assume that $U_{n-1}(i) = U'_{n-1}(i) + f_{n-1}(i)$. Then we have

$$\begin{aligned} U_n(i) &= {}^1\sigma_n^*(U_{n-1}(i)) \\ &= {}^1\sigma_n^*(U'_{n-1}(i) + f_{n-1}(i)) && \text{from the induction hypothesis} \\ &= {}^1\sigma_n^*(U'_{n-1}(i)) + A_{(a_n, \varepsilon_n)}^{-1} f_{n-1}(i) && \text{from (4.6)} \\ &= U'_n(i) + f_n(i) && \text{from (5.12)} \end{aligned}$$

and the lemma follows. $\square$

Taking (5.3), (5.10) and Lemma 5.2 into account, we define the domain exchange transformation.

Definition 5.1 *We define the mapping* $\mathcal{Q}_n$ *from* $\mathcal{R}(U'_n)$ *to* $\mathcal{R}(U_n)$ *and the transformation* Q_n *on* D_n *as follows:*

$$\begin{aligned} \mathcal{Q}_n(x) &= x + f_n(i) && \text{if } x \in \mathcal{R}(U'_n(i)) \\ Q_n(x) &= x + \pi_n f_n(i) && \text{if } x \in D'_n(i). \end{aligned}$$

$\mathcal{Q}_n$ *and* Q_n *are called domain exchange transformations.*

The next lemma shows that U_n, U'_n, and D_n give periodic tiling.

Lemma 5.3 *It follows that*

$$S_n = \bigsqcup_{z \in \mathcal{L}_n} (z + \mathcal{R}(U_n)), \quad S'_n = \bigsqcup_{z \in \mathcal{L}_n} (z + \mathcal{R}(U'_n)),$$

and

$$\mathbb{R}^d = \bigsqcup_{z \in \pi_n \mathcal{L}_n} (z + D_n).$$

PROOF. It is trivial to show that

$$S_0 = \bigsqcup_{z \in \mathcal{L}_0} (z + \mathcal{R}(U_0)) \quad \text{and} \quad S'_0 = \bigsqcup_{z \in \mathcal{L}_0} (z + \mathcal{R}(U'_0)).$$

Then from the definition of ${}^1\sigma^*_n$, (4.5), and Lemma 4.3, the lemma holds. □

In the next lemma, we see that the domain exchange transformation is an irrational rotation on $\mathbb{R}^d/\pi_n\mathcal{L}_n$. Figure 5.4 and Figure 5.5 show examples of D_7 and D'_7 for the same α as in the preceding figures. In these figures, large hexagons drawn with dashed lines denote $\mathbb{R}^2/\pi_7\mathcal{L}_7$.

Lemma 5.4 *For all* $x \in D_n$*,*

$$Q_n(x) = x + \pi_n f_n(0) \pmod{\pi_n \mathcal{L}_n}.$$

PROOF. We see that

$$\begin{aligned} \mathcal{L}_n &= M_n(\alpha)^{-1}\mathcal{L}_0 \\ &= \begin{pmatrix} f_n(0) & f_n(1) & \cdots & f_n(d) \end{pmatrix} \sum_{i=1}^{d} \mathbb{Z}\,(\mathbf{e}_i - \mathbf{e}_0) \\ &= \sum_{i=1}^{d} \mathbb{Z}(f_n(i) - f_n(0)) \end{aligned}$$

holds from (4.5) and (5.12). Then,

$$\pi_n f_n(0) = \pi_n f_n(1) = \cdots = \pi_n f_n(d) \pmod{\pi_n \mathcal{L}_n}.$$

From Definition 5.1 and Lemma 5.3, the lemma follows. □

Lemma 5.5

$$-B_n(\alpha) Q_n B_n(\alpha)^{-1} = R_\alpha.$$

PROOF. From Lemma 5.4, we see that $Q_n = F_\alpha \pmod{\pi_n \mathcal{L}_n}$. Then, from Lemma 5.3, it is enough to show that

$$B_n(\alpha)\pi_n \mathcal{L}_n = L_\alpha \tag{5.13}$$

and

$$B_n(\alpha)\pi_n f_n(0) = -{}^t\alpha \tag{5.14}$$

hold for all $n \in \mathbb{Z}_{\geq 0}$. In the case where $n = 0$, we have (5.13) and (5.14), from Definition 1.1 and the definition of π_n. We assume that

$$B_{n-1}(\alpha)\pi_{n-1} f_{n-1}(i) = \pi_0 f_0(i) \tag{5.15}$$

holds. From Definition 2.3 and Proposition 2.1, we have

$$\begin{pmatrix} 1 \\ {}^t\alpha_{n-1} \end{pmatrix} = \alpha^1_{n-1} M(\alpha_{n-1}) \begin{pmatrix} 1 \\ {}^t\alpha_n \end{pmatrix}. \tag{5.16}$$

From Definition 2.4,

$$B(\alpha_{n-1})\mathbf{e}_i = \begin{cases} \mathbf{e}_{i+1} & \text{if} \quad i = 1, \ldots, \varepsilon_n - 1 \\ -{}^t\alpha_{n-1} & \text{if} \quad i = \varepsilon_n \\ \mathbf{e}_i, & \text{if} \quad i = \varepsilon_n + 1, \ldots, d, \end{cases} \tag{5.17}$$

holds. From (5.12), we also have

$$A^{-1}_{(a_n,\varepsilon_n)} \begin{pmatrix} f_{n-1}(0) & f_{n-1}(1) & \cdots & f_{n-1}(d) \end{pmatrix} = \begin{pmatrix} f_n(0) & f_n(1) & \cdots & f_n(d) \end{pmatrix}. \tag{5.18}$$

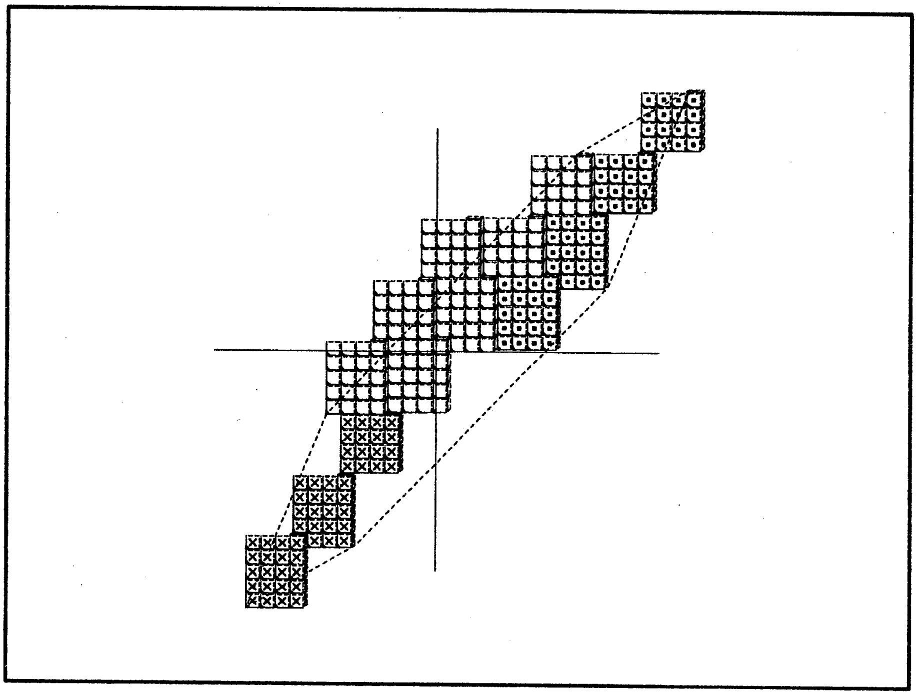

Fig. 5.4. D_7

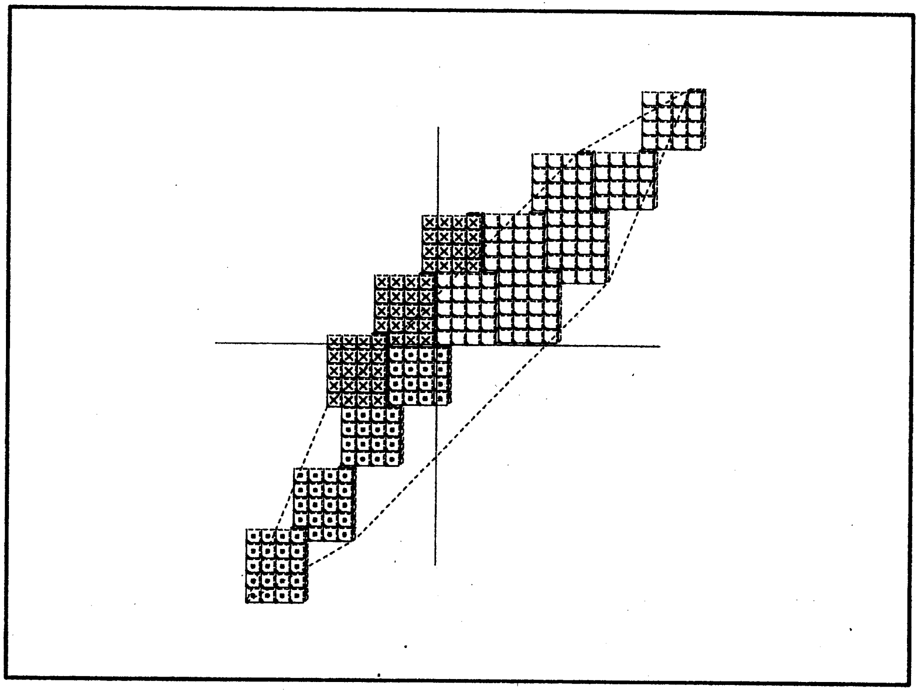

Fig. 5.5. D_7'

By using (5.16), (5.17) and (5.18), we see that

(5.19)

$$
\begin{aligned}
&B(\alpha_{n-1})\pi_n f_n(i) \\
&= B(\alpha_{n-1})\begin{pmatrix}-{}^t\alpha_n & I_d\end{pmatrix} f_n(i) \\
&= B(\alpha_{n-1})\Big(f_{n-1}(i)^1(-{}^t\alpha_n) + f_{n-1}(i)^2\mathbf{e}_1 + \cdots + f_{n-1}(i)^{\varepsilon_n}\mathbf{e}_{\varepsilon_n-1} \\
&\quad + \left(f_{n-1}(i)^0 - a_n f_{n-1}(i)^1\right)\mathbf{e}_{\varepsilon_n} \\
&\quad + f_{n-1}(i)^{\varepsilon_n+1}\mathbf{e}_{\varepsilon_n+1} + \cdots + f_{n-1}(i)^d\mathbf{e}_d\Big) \\
&= B(\alpha_{n-1})\Bigg(f_{n-1}(i)^1\Bigg(-\frac{1}{\alpha_{n-1}^1}{}^t\big(\alpha_{n-1}^2, \ldots, \alpha_{n-1}^{\varepsilon_n}, 1 - a_n\alpha_{n-1}^1, \\
&\quad \alpha_{n-1}^{\varepsilon_n+1}, \ldots, \alpha_{n-1}^d\big)\Bigg) + f_{n-1}(i)^2\mathbf{e}_1 + \cdots + f_{n-1}(i)^{\varepsilon_n}\mathbf{e}_{\varepsilon_n-1} \\
&\quad + (f_{n-1}(i)^0 - a_n f_{n-1}(i)^1)\mathbf{e}_{\varepsilon_n} \\
&\quad + f_{n-1}(i)^{\varepsilon_n+1}\mathbf{e}_{\varepsilon_n+1} + \cdots + f_{n-1}(i)^d\mathbf{e}_d\Bigg) \\
&= f_{n-1}(i)^1 B(\alpha_{n-1})\left(-\frac{1}{\alpha_{n-1}^1}{}^t\big(\alpha_{n-1}^2, \ldots, \alpha_{n-1}^{\varepsilon_n}, 1, \alpha_{n-1}^{\varepsilon_n+1}, \ldots, \alpha_{n-1}^d\big)\right) \\
&\quad + f_{n-1}(i)^2 B(\alpha_{n-1})\mathbf{e}_1 + \cdots + f_{n-1}(i)^{\varepsilon_n} B(\alpha_{n-1})\mathbf{e}_{\varepsilon_n-1} \\
&\quad + f_{n-1}(i)^0 B(\alpha_{n-1})\mathbf{e}_{\varepsilon_n} + f_{n-1}(i)^{\varepsilon_n+1} B(\alpha_{n-1})\mathbf{e}_{\varepsilon_n+1} + \cdots + f_{n-1}(i)^d\mathbf{e}_d \\
&= f_{n-1}(i)^1\mathbf{e}_1 + \cdots + f_{n-1}(i)^{\varepsilon_n}\mathbf{e}_{\varepsilon_n} + f_{n-1}(i)^0(-{}^t\alpha_{n-1}) \\
&\quad + f_{n-1}(i)^{\varepsilon_n+1}\mathbf{e}_{\varepsilon_n+1} + \cdots + f_{n-1}(i)^d\mathbf{e}_d \\
&= \begin{pmatrix}-{}^t\alpha_{n-1} & I_d\end{pmatrix} f_{n-1}(i) \\
&= \pi_{n-1} f_{n-1}(i)
\end{aligned}
$$

holds for all $i \in \mathcal{A}$. Multiplying the equality (5.19) by $B_{n-1}(\alpha)$ from the left, we see that

$$B_n(\alpha)\pi_n f_n(i) = \pi_0 f_0(i)$$

holds from (5.15). (5.13) is proved in an analogous way. □

We now have the following theorem, which gives the geometrical characterization of the distribution of the Kronecker sequence. This theorem implies that elements of the Kronecker sequence reside in the lattice that is the projection of the stepped surface. Figures 5.6 and 5.7 show $B_4(\alpha)\pi_4 U_4$ and $B_4(\alpha)\pi_4(\mathcal{R}(U_4) \cap \mathbb{Z}^3)$ respectively where α is the same as in the preceding figures. In these figures, large hexagons denote $\mathbb{R}^2/(B_4(\alpha)\pi_4\mathcal{L}_4) = \mathbb{R}^2/L_\alpha$. Let H be a polyhedron which represents $\mathbb{R}^d/L_\alpha$.

Theorem 5.1 *For all $n \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned}\{K_\alpha(k)\}_{k=1}^{\sum_{j=0}^{d} l_n(\alpha;j)} &= -B_n(\alpha)\pi_n\left(\mathcal{R}(U_n)\cap \mathbb{Z}^{d+1}\right) \pmod{L_\alpha}\\ &= H \cap \left(-B_n(\alpha)\pi_n\left(S_n \cap \mathbb{Z}^{d+1}\right)\right)\end{aligned}$$

holds.

PROOF. We abbreviate $l_n(\alpha; j)$ to l_n^j in the following. From Definition 1.2 and Lemma 5.5, it is enough to show that

$$\left\{Q_n^k(0)\right\}_{k=1}^{\sum_{j=0}^{d} l_n^j} = \pi_n\left(\mathcal{R}(U_n)\cap \mathbb{Z}^{d+1}\right) \tag{5.20}$$

holds.

For $V \in \mathcal{S}_n$, we define $\mathcal{C}_n(V) = \{(x,i^*) \mid (x,i^*) \prec V\}$ and $\mathcal{C}_n(V;i) = \{(x,j^*) \in \mathcal{C}_n(V) \mid j = i\}$. From (4.6), Lemma 4.3, and Definition 2.3, we have

$$\#\mathcal{C}_n(U_n;j) = l_n^j \quad \text{for all } j \in \mathcal{A}. \tag{5.21}$$

We consider the orbit of $U'(i)$, $(i \in \mathcal{A})$ by the transformation $\mathcal{Q}_n$. Taking account of Lemmas 5.1, 5.2, and 5.4, we define a set of elements of $\mathcal{C}_n(U_n;i) \cup \{U'(i)\}$ and $m_i \in \mathbb{N}$ as follows:

$$V_j^i = \begin{cases} U'(i) & \text{if } j = 0,\\ V_j^i = \mathcal{Q}_n(V_{j-1}^i) & \text{if } V_{j-1}^i \in (\mathcal{C}_n(U_n;i)\setminus\{U(i)\})\cup\{U'(i)\}\\ & \text{and } \mathcal{Q}_n(V_{j-1}^i)\cap\{V_0^i,\dots,V_{j-1}^i\} = \emptyset,\\ \text{undefined} & \text{otherwise,}\end{cases}$$
$$m_i = \max\{j \in \mathbb{Z}_{\geq 0} \mid V_j^i \text{ exists}\}.$$

If we assume that there exists some $j \in \mathbb{N}$ that satisfies $j < l_n^i - 1$ and $\mathcal{Q}_n(V_j^i) = U(i)$, then $m_i = j$ and, from (5.21), there exists a non-empty subset $\mathcal{B}$ of $\mathcal{C}_n(U_n;i)\setminus\{U(i)\}\setminus\{V_0^i,\dots,V_m^i\}$ that satisfies $\mathcal{Q}_n\mathcal{B} = \mathcal{B}$. This implies that

$$Q_n\left(\pi_n \bigcup_{V\in\mathcal{B}} \mathcal{R}(V)\right) = \pi_n \bigcup_{V\in\mathcal{B}} \mathcal{R}(V),$$

and contradicts the fact that irrational rotation Q_n is ergodic. If we assume that there exists some $j, j' \in \mathbb{N}$ that satisfies $j' < j < l_n^i - 1$ and $\mathcal{Q}_n(V_j^i) = V_{j'}^i$, then $\mathcal{Q}_n(\{V_{j'}^i,\dots,V_j^i\}) = \{V_{j'}^i,\dots,V_j^i\}$. This is also a contradiction. Thus,

$$\begin{aligned}\{U'(i), \mathcal{Q}_n(U'(i)),\dots,\mathcal{Q}_n^{l_n^i-1}(U'(i))\}&\\ = (\mathcal{C}_n(U_n;i)\setminus\{U(i)\})&\cup\{U'(i)\}\end{aligned} \tag{5.22a}$$

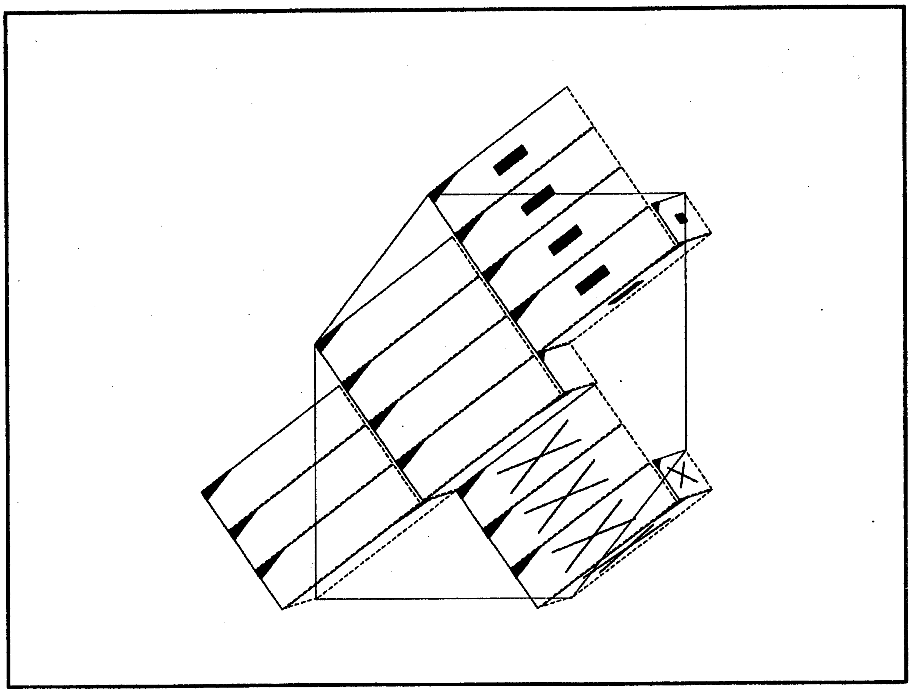

Fig. 5.6. $B_4(\alpha, \beta)\pi_4 U_4$

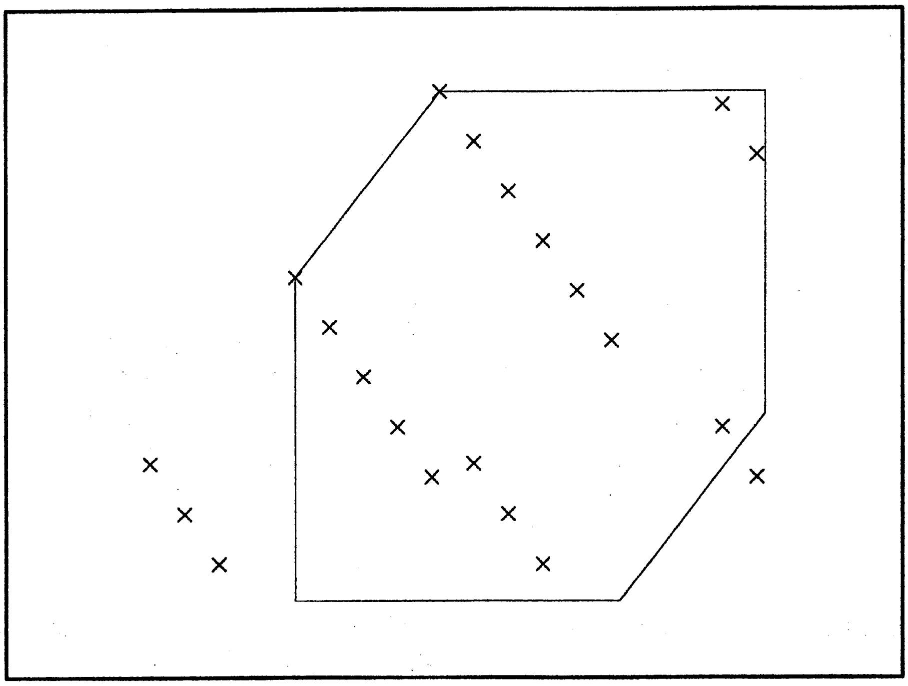

Fig. 5.7. $B_4(\alpha, \beta)\pi_4(\mathcal{R}(U_4) \cap \mathbb{Z}^3)$

and

$$(5.22b) \qquad Q_n^{l_n^i}(U'(i)) = U(i).$$

hold. From Definition 4.5

$$(5.23a) \qquad \mathcal{R}(U(j)) \cap \mathbb{Z}^{d+1} = \mathcal{R}(U'(j+1)) \cap \mathbb{Z}^{d+1} \quad \text{for } j \in \mathcal{A} \setminus \{d\}$$

and

$$(5.23b) \qquad \{0\} = \mathcal{R}(U'(0)) \cap \mathbb{Z}^{d+1}$$

hold. From (5.22) and (5.23), we have

$$(5.24) \qquad Q_n^{l_n^0+\cdots+l_n^i}(0) \in \pi_n\left(\mathcal{R}(U(i)) \cap \mathbb{Z}^{d+1}\right) \quad \text{for } i \in \mathcal{A}.$$

Then we have (5.20). □

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